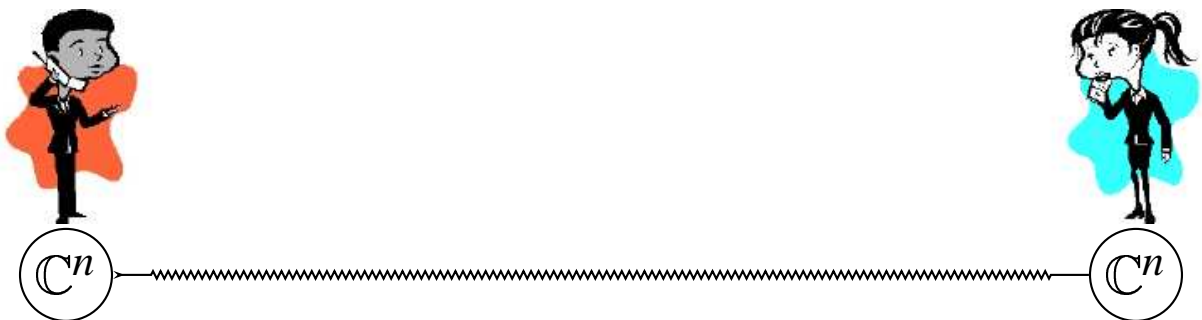


Distinguishing Separable and Entangled States

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Entanglement of Mixed States

Composite quantum systems

- System **A** in ρ_A & System **B** in ρ_B
- ↪ Composite system **A** \otimes **B** in state $\rho_A \otimes \rho_B$.
- Mixing locally prepared states
↪ unentangled.

Definition

$\rho \in T_1^+(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is **separable** if

$$\rho = \sum_k \omega_k \rho_k^{(A)} \otimes \rho_k^{(B)}$$

where $\forall k: \omega_k \geq 0, \rho_k^{(A,B)} \in T_1^+(\mathbb{C}^{d_{A,B}})$.

- in general difficult to decide ↪ separability criteria

Peres Criterion

Partial Transpose

$$\langle ij|\rho^{T_2}|kl\rangle \equiv \langle il|\rho|kj\rangle.$$

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \rightarrow \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{22} & \rho_{14} & \rho_{24} \\ \rho_{31} & \rho_{41} & \rho_{33} & \rho_{43} \\ \rho_{32} & \rho_{42} & \rho_{34} & \rho_{44} \end{pmatrix}$$

- T is positive, but not completely positive.
- T is contractive, not completely contractive.

Peres PPT Criterion

If $\rho \in T(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is separable, then

$$\rho^{T_2} \geq 0.$$

$\rho \in T(\mathbb{C}^2 \otimes \mathbb{C}^{2;3})$ is separable iff $\rho^{T_2} \geq 0$.

Positive maps approach

Theorem [Horodecki³]

ρ is separable iff $\mathbb{1}_A \otimes \Lambda_B(\rho) \geq 0$ for all positive linear $\Lambda_B : \mathbb{T}(\mathbb{C}^n) \rightarrow \mathbb{T}(\mathbb{C}^n)$.

Separability Criteria

If $\mathbb{1}_A \otimes \Lambda_B(\rho) \not\geq 0$, then ρ is entangled.

- Peres criterion: $\Lambda_B(\sigma) = \sigma^T$.
- Reduction criterion: $\Lambda_B = \text{tr}(\sigma)\mathbb{1} - \sigma$

Entanglement Witnesses

ρ is separable iff $\text{tr}(W\rho) \geq 0$

for any operator W with $\text{tr}(W\rho_1 \otimes \rho_2) \geq 0$.

- W with $\text{tr}(W\rho) < 0$ is called an *entanglement witness* for ρ .

Bound entanglement

Distillability $\rho \xrightarrow{\text{LOCC}} |\psi_+\rangle\langle\psi_+|$.

Bound entangled states

- ρ entangled and $\rho \not\xrightarrow{\text{LOCC}} |\psi_+\rangle\langle\psi_+|$.

Proposition ρ distillable $\Rightarrow \rho^{T_2} \not\geq 0$.

Proof

$$\checkmark \Lambda_{\text{LOCC}}(\sigma) = \sum_i A_i \otimes B_i \sigma A_i^\dagger \otimes B_i^\dagger$$

$$\checkmark (A \otimes B \rho A^\dagger \otimes B^\dagger)^{T_2} = A \otimes B^* \rho^{T_2} A^\dagger \otimes B^T.$$

$\curvearrowright \Lambda_{\text{LOCC}}$ preserves PPT.

Cross Norms

Definition

Two normed spaces: $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$

↪ Norm $\|\cdot\|_{AB}$ on $A \otimes_{\text{alg}} B$ is called

- **cross** if $\|\sigma_A \otimes \sigma_B\|_{AB} = \|\sigma_A\|_A \|\sigma_B\|_B$
- **subcross** if $\|\sigma_A \otimes \sigma_B\|_{AB} \leq \|\sigma_A\|_A \|\sigma_B\|_B$.

Quantum Information

$$A, B = \mathbb{T}(\mathbb{C}^d), \|\cdot\|_{A,B} = \|\cdot\|_1$$

$$A \otimes_{\text{alg}} B = \mathbb{T}(\mathbb{C}^d \otimes \mathbb{C}^d) \cong d^2 \times d^2 \text{ matrices.}$$



Trace norm: $\sigma \in \mathbb{T}(\mathbb{C}^d)$ with singular values $\lambda_1, \dots, \lambda_d$.

$$\|\sigma\|_1 := \sum_{i=1}^d |\lambda_i|.$$

Hilbert-Schmidt norm: $\|\sigma\|_2 := \sqrt{\sum_{i=1}^d |\lambda_i|^2}$.

Greatest Cross Norm & Separability

The greatest cross norm

$$\|\sigma\|_{\gamma} := \inf \left\{ \sum_{i=1}^k \|\sigma_i^A\|_1 \|\sigma_i^B\|_1 \mid \sigma = \sum_{i=1}^k \sigma_i^A \otimes \sigma_i^B \right\}$$

Theorem ρ is separable iff

- $\|\rho\|_{\gamma} = 1$.
- $\|\rho\|_{sc} \leq 1$ for all subcross norms $\|\cdot\|_{sc}$.

Greatest ? Let $\|\cdot\|_{sc}$ be a subcross norm

✓ $\|\sigma\|_{sc} \leq \sum_{i=1}^k \|\sigma_i^A\|_1 \|\sigma_i^B\|_1$ (subadditivity)

✓ take infimum on rhs. \square

Computable Cross Norm

The computable subcross norm

$$\|\sigma\|_{\tau} := \inf \left\{ \sum_{i=1}^k \|\sigma_i^A\|_2 \|\sigma_i^B\|_2 \mid \sigma = \sum_{i=1}^k \sigma_i^A \otimes \sigma_i^B \right\}$$

- $\|\sigma\|_{\tau} \leq \|\sigma\|_{\gamma} \Rightarrow \rho$ separable $\Leftrightarrow \|\rho\|_{\tau} \leq 1$.

Computable ?

$$\|O\|_1 = \inf \left\{ \sum_{i=1}^k \|\psi_i\| \|\phi_i\| \mid O = \sum_{i=1}^k |\psi_i\rangle \langle \phi_i| \right\}$$

\Leftrightarrow **Theorem** $\|\sigma\|_{\tau} = \|\mathfrak{A}(\sigma)\|_1$

- $\rho = \sum_i r_i E_i \otimes F_i \iff \mathfrak{A}(\rho) = \sum_i r_i |E_i\rangle \langle F_i^*|,$

$$\rho \in \mathcal{T}(\mathbb{C}^d \otimes \mathbb{C}^d) \iff \mathfrak{A}(\rho) \in \mathcal{L}(\mathcal{T}(\mathbb{C}^d), \mathcal{T}(\mathbb{C}^d))$$

- $$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \iff \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{21} & \rho_{22} \\ \rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{41} & \rho_{42} \\ \rho_{33} & \rho_{34} & \rho_{43} & \rho_{44} \end{pmatrix}$$

CCN criterion and bound entanglement

Example $\mathbb{C}^3 \otimes \mathbb{C}^3: 2 < \alpha \leq 5,$

$$\rho_\alpha := \frac{2}{7} \left| \Psi_{(3)}^+ \right\rangle \left\langle \Psi_{(3)}^+ \right| + \frac{\alpha}{7} \sigma_+ + \frac{5-\alpha}{7} \sigma_-$$

$$\sigma_+ \equiv \frac{1}{3} (|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|)$$

$$\sigma_- \equiv \frac{1}{3} (|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|).$$

$$\|\mathfrak{A}(\rho_\alpha)\|_1 = \frac{19}{21} + \frac{2}{21} \sqrt{19 - 15\alpha + 3\alpha^2}$$

$\curvearrowright \|\mathfrak{A}(\rho_\alpha)\|_1 > 1$, for $3 < \alpha \leq 5$.

- $\|\rho^{\text{T}_2}\|_1 \leq 1$ for $3 < \alpha \leq 4$.

$\curvearrowright \rho_\alpha$ bound entangled for $3 < \alpha \leq 4$.

Genuine Tripartite Entanglement

Three qubit state

$$\rho_{ABC} = \frac{1}{8} \left(\mathbb{1} - \sum_{i=1}^4 |\psi_i\rangle\langle\psi_i| \right)$$

- $\{|\psi_i\rangle\} = \{|0, 1, +\rangle, |1, +, 0\rangle, |+, 1, 0\rangle, |-, -, -\rangle\}$
with $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$
= unextendable product basis.

- ρ_{ABC} is bi-separable and bound entangled.

- based on range analysis

[Bennett et al. PRL **53** (1999), 5385].

Cross Norm Criterion

$$\|(\mathbb{1} \otimes \mathcal{A})\rho_{ABC}\|_1 \simeq 1.08649$$

- ↪ ρ_{ABC} is entangled.

[source: M., P., and R. Horodecki, quant-ph/0206008]

Linear contractions approach

Theorem

ρ is separable iff

- $\mathbb{1}_A \otimes \Lambda_B(\rho) \geq 0$
for all positive trace-preserving linear $\Lambda_B : \mathbb{T}(\mathbb{C}^d) \rightarrow \mathbb{T}(\mathbb{C}^{2d})$.
- $\|\mathbb{1}_A \otimes \Lambda_B(\rho)\|_1 \leq 1$
for all positive trace preserving linear $\Lambda_B : \mathbb{T}(\mathbb{C}^d) \rightarrow \mathbb{T}(\mathbb{C}^{2d})$.
- $\|\Lambda(\rho)\|_1 \leq 1$
for all linear Λ with $\|\Lambda(\rho_1 \otimes \rho_2)\|_1 \leq 1$.
- $\|\rho\|_{sc} \leq 1$
for all subcross norms $\|\cdot\|_{sc}$.