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Physics Letters A 335 (2005) 11–19
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Perfect correlations between noncommuting observables

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Received 9 August 2004; received in revised form 4 December 2004; accepted 9 December 2004
Available online 22 December 2004
Communicated by P.R. Holland

Abstract

The problem as to when two noncommuting observables are considered to have the same value arises commonly, but shows a nontrivial difficulty. Here, an answer is given by establishing the notion of perfect correlations between noncommuting observables and applied to obtain a criterion for precise measurements of a given observable in a given state.

PACS: 03.65.Ta; 03.67.-a

Keywords: Quantum correlations; Perfect correlations; Measurements; Observables

1. Introduction

In quantum mechanics, we cannot predict a definite value of a given observable generally, and it is sometimes stressed that quantum mechanics does not speak of the value of an observable in a single event, but only speaks of the average value over a large number of events. However, the quantum correlation definitely describes relations of values of observables in a single event as typically in the EPR correlation, where we cannot predict a definite value of the momentum or the position of each particle from an EPR pair, whereas we can definitely predict the total momentum and the distance of the pair, and thereby we have a definite one-to-one correspondence between the values of their momenta to be obtained from their joint measurements or between the values of their positions.

In this Letter, we shall investigate one of the most fundamental aspects of quantum correlations; that is, we shall consider the general problem as to when two observables X and Y in a quantum system can be considered to "have the same value", in a given state, in the sense suggested above. It should be stressed that when we use this expression, we do not intend to make any assumptions as to whether a definite value exists prior to the measurement; such a question is a matter of the interpretation of quantum mechanics and we do not enter into it. Rather, we choose to define what it means "to have the same value" in terms of perfect correlations as the ones described above, meaning that, if the two observables are jointly measured, one is guar-

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2004.12.003

anteed to obtain the same value for both. As we shall explain below, the question of when two observables X and Y "have the same value" arises when one asks if the time evolution changes the given observable and if an indirect measurement consisting of the measuring interaction and the meter measurement is considered to precisely measure the given observable.

For two classical random variables X and Y, it is well accepted that X and Y have the same value if and only if X and Y are perfectly correlated, or equivalently the joint probability of obtaining different values of X and Y vanishes. Thus, we can immediately generalize this notion to pairs of commuting observables based on the well-defined joint probability distribution of commuting observables, so that two commuting observables are considered to have the same value in the given state if and only if they are perfectly correlated. However, two operators are not necessarily commuting, and the generalization of the notion of perfect correlation to noncommuting observables should be strongly demanded, whereas no serious investigations have been done. This Letter introduces the notion of perfect correlations between arbitrary two observables, and characterizes it by various statistical notions in quantum mechanics. As a result, the above problems are shown to be answered by simple and well-founded conditions in the standard formalism of quantum mechanics.

2. Difficulties in the notion of perfection correlation

Let A be an observable of a quantum system in a state ψ at the origin of time. Then, it is a fundamental question to ask whether the observable A is unchanged or changed between two times t_1 and t_2. Let $A(t)$ be the Heisenberg operator at time t corresponding to the observable A. If the question is asked independent of the system state ψ, the answer is that A is unchanged if and only if $A(t_1) = A(t_2)$. However, the question depending on the system state shows a nontrivial difficulty.

Let $D_A = A(t_2) - A(t_1)$ be the increment of A from time t_1 to t_2. Then, it is natural to expect that the value of the observable A is unchanged between two times t_1 and t_2 in the system state ψ if and only if the state ψ is an eigenstate of D_A with eigenvalue 0, i.e., $D_A\psi = 0$, or equivalently

$$A(t_1)\psi = A(t_2)\psi. \tag{1}$$

This means that the increment D_A has the definite value zero in the state ψ. However, the above characterization is unexpectedly not true in general. For example, let $A(t_1)$ and $A(t_2)$ be two 4×4 matrices such that

$$A(t_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A(t_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

with time evolution operator $U(t_2, t_1)$ and the state ψ such that

$$U(t_2, t_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, we have $A(t_1)\psi = A(t_2)\psi$, and hence the first and the second moments of A are unchanged, i.e., $\langle\psi|A(t_1)|\psi\rangle = \langle\psi|A(t_2)|\psi\rangle = 1$ and $\langle\psi|A(t_1)^2|\psi\rangle = \langle\psi|A(t_2)^2|\psi\rangle = 2$. However, we have $\langle\psi|A(t_1)^3|\psi\rangle = 4$ but $\langle\psi|A(t_2)^3|\psi\rangle = 3$. Thus, the third moment of A is changed from time t_1 to t_2, so that the observable A is considered to have been changed in this time interval.

On the other hand, the requirement that $A(t_1)$ and $A(t_2)$ should have the same probability distribution in the state ψ is a necessary but not sufficient condition, since there are cases where $A(t_1)$ and $A(t_2)$ have the same probability distribution but they are statistically independent. Specifically, suppose that ψ is the product state of two copies of a state ϕ, i.e., $\psi = \phi \otimes \phi$, and that there is an observable B such that $A(t_1) = B \otimes I$ and $A(t_2) = I \otimes B$. In this case, $A(t_1)$ and $A(t_2)$ have the same probability distribution, but they are statistically independent in the case where ϕ is not an eigenstate of B. In fact, $D_A\psi \neq 0$ if and only if ϕ is not an eigenstate of B. Thus, in this case, we cannot judge that the observable A has been unchanged.

3. Perfect correlation in measurement

The notion of perfect correlation is not restricted to the problem on the Heisenberg time evolution, but also has broad applications in foundations on quantum mechanics [1] and quantum information theory [2]. Among them, another problem concerns the notion of measurement. Any measurement has two not necessarily commuting observables, one of which is the observable to be measured and the other is the meter observable after the measuring interaction [3,4]. A fundamental question as to when the given observable is precisely measured in a given state has remained open. However, this is obviously related to the perfect correlation between the measured observable and the meter observable. This Letter will solve this fundamental problem by establishing the general notion of perfect correlations between noncommuting observables.

Every measurement can be modeled by a process of indirect measurement described by the measuring interaction between the measured object and the measuring apparatus followed by a subsequent observation of the meter observable in the apparatus [2–8]. Let **S** be the object and **A** the apparatus. Then, in order to measure the value of an observable A in **S** at time t, the time of the measurement, the observer actually observes the value of the meter observable M in **A** at time $t + \Delta t$, where the measuring interaction is supposed to be turned on from time t to $t + \Delta t$. Thus, in order to measure the observable $A(t)$, the indirect measurement actually observes the observable $M(t + \Delta t)$.

A fundamental problem is to determine what condition ensures that this measurement successfully measures the value of the observable A at time t. If we have a satisfactory notion of perfect correlation, we can readily answer this question by stating that the indirect measurement successfully measures the observable A at time t if and only if $A(t)$ and $M(t + \Delta t)$ are perfectly correlated. However, since $A(t)$ and $M(t + \Delta t)$ are not necessarily commuting, the above question has not been answered generally.

Instead, the conventional approach has questioned what observable is measured by the above indirect measurement independent of the input state. Let ψ be the state of **S** at time t and ξ the state of **A** at time t. We assume that the apparatus **A** is always prepared in the fixed state ξ at the time of the measurement, while the object **S** is in an arbitrary state ψ. Then, the indirect measurement measures the observable A at time t if and only if the two observables $A(t)$ and $M(t + \Delta t)$ have the same probability distribution for any state ψ [2–4,6–8].

Since the above definition of measurement of the observable independent of the input state does not explicitly require that $A(t)$ and $M(t + \Delta t)$ have the same value unless the measurement is carried out in an eigenstate of $A(t)$, it is not immediately obvious whether the value randomly obtained by observing $M(t + \Delta t)$ would actually correspond in any way to the value one would obtain, in the same situation, by an alternative (indirect or direct) measurement of $A(t)$. Yet, there is something unsatisfying about the possibility that, for any state not an eigenstate of $A(t)$, the two operators $A(t)$ and $M(t + \Delta t)$ might just represent independent random variables that just happen to have the same distribution. One would certainly like to think that, in a precise measurement, these two operators should "have the same value"—in the sense defined in the Introduction—even under conditions when this value may not be a definite quantity prior to the measurement.

Indeed, the experimenter reads the value of the meter M at time $t + \Delta t$ and records that the same value was taken by A at time t; however, there might be a possibility that another experimenter would obtain a different value of A at time t from another apparatus. As above, it has not been ensured that this is not the case.

In order to solve the above problem, this Letter introduces the notion of perfect correlations between arbitrary two observables in any state, and characterizes it by various statistical notions in quantum mechanics. As a result, the above problem is affirmatively answered by simple and well-founded conditions in the standard formalism of quantum mechanics. In particular, we shall establish a simple condition for the measured observable $A(t)$ and the meter observable $M(t + \Delta t)$ to be perfectly correlated in a given state ψ, and show that the conventional definition implies that the measured observable $A(t)$ and the meter observable $M(t + \Delta t)$ are actually perfectly correlated in any state ψ. Thus, we shall conclude that the measured value from the meter observable after the measuring interaction is not produced by the interaction, but actually reproduces the value of the measured observable before the interaction.

4. Definition of perfect correlations

Let X, Y be two observables in a quantum system **S** described by a Hilbert space $\mathcal{H}$. For simplicity, in this Letter we assume that $\mathcal{H}$ is finite-dimensional. The spectral projection $E^X(x)$ of X for any $x \in \mathbf{R}$ is generally defined to be the projection operator of $\mathcal{H}$ onto the subspace $\{\psi \in \mathcal{H} \mid X\psi = x\psi\}$. If X and Y commute, their joint probability distribution in an arbitrary state ψ is defined by

$$\Pr\{X = x, Y = y \| \psi\} = \langle\psi|E^X(x)E^Y(y)|\psi\rangle. \tag{2}$$

The above probability distribution is operationally interpreted as the joint probability distribution of the measured values of X and Y in the simultaneous measurement of X and Y. In general, we say that X and Y are *jointly distributed* in state ψ, if

$$\langle\psi|E^X(x)E^Y(y)|\psi\rangle \geqslant 0 \tag{3}$$

for any $x, y \in \mathbf{R}$. In this case, we have

$$\langle\psi|E^X(x)E^Y(y)|\psi\rangle = \langle\psi|E^Y(y)E^X(x)|\psi\rangle. \tag{4}$$

Then, for any function $F(x, y) = \sum_{j,k} f_j(x)g_k(y)$ we have

$$\begin{aligned}
&\sum_{x,y} F(x, y)\langle\psi|E^X(x)E^Y(y)|\psi\rangle \\
&\quad = \langle\psi|\sum_{j,k} f_j(X)g_k(Y)|\psi\rangle \\
&\quad = \langle\psi|\sum_{j,k} g_k(Y)f_j(X)|\psi\rangle.
\end{aligned} \tag{5}$$

We say that X and Y are *perfectly correlated* in state ψ, if

$$\langle\psi|E^X(x)E^Y(y)|\psi\rangle = 0 \tag{6}$$

for any $x, y \in \mathbf{R}$ with $x \neq y$. It is obvious that perfectly correlated observables are jointly distributed. Since $\langle\psi|E^X(x)|\psi\rangle = \sum_y \langle\psi|E^X(x)E^Y(y)|\psi\rangle$, the above condition is equivalent to the relation

$$\langle\psi|E^X(x)E^Y(y)|\psi\rangle = \delta_{x,y}\langle\psi|E^X(x)|\psi\rangle \tag{7}$$

for any $x, y \in \mathbf{R}$, where $\delta_{x,y}$ stands for Kronecker's delta. If X and Y are commuting, the above definition reduces to the usual one that means that in the simultaneous measurement of X and Y the joint probability of the results $X = x$ and $Y = y$ vanishes, if $x \neq y$. We shall show that a pair of observables X, Y perfectly correlated in a state ψ are considered to be simultaneously measurable in the state ψ and that their outcomes always coincide each other.

We say that two observables X and Y are *identically distributed* in state ψ, if $\langle\psi|E^X(x)|\psi\rangle = \langle\psi|E^Y(x)|\psi\rangle$ for all $x \in \mathbf{R}$. It follows easily from Eq. (7) that perfectly correlated observables are identically distributed. However, it is also obvious that the converse is not true even for commuting observables.

5. Root mean square of difference

Suppose that X and Y are perfectly correlated in ψ. Then, intuitively speaking, they have the same value, even though both of them are random. Thus, it is expected that the difference $X - Y$ definitely has the value zero, or equivalently ψ is an eigenstate of $X - Y$ with eigenvalue 0, i.e., $X\psi = Y\psi$. In order to prove this property from our definition, we consider the distance $\|X\psi - Y\psi\|$ between $X\psi$ and $Y\psi$. Obviously, $\|X\psi - Y\psi\| = 0$ if and only if $X\psi = Y\psi$. We generally have

$$\begin{aligned}
&\|X\psi - Y\psi\|^2 \\
&\quad = \left\| \sum_x xE^X(x)\psi - \sum_y yE^Y(y)\psi \right\|^2 \\
&\quad = \sum_{x,y} (x - y)^2 \Re\langle\psi|E^X(x)E^Y(y)|\psi\rangle.
\end{aligned}$$

Thus, if X and Y are jointly distributed in state ψ, we have

$$\|X\psi - Y\psi\|^2 = \sum_{x,y} (x - y)^2 \langle\psi|E^X(x)E^Y(y)|\psi\rangle. \tag{8}$$

Suppose that X and Y are perfectly correlated in state ψ. Then, we have

$$\begin{aligned}
&\sum_{x,y} (x - y)^2 \langle\psi|E^X(x)E^Y(y)|\psi\rangle \\
&\quad = \sum_{x,y} (x - y)^2 \delta_{x,y}\langle\psi|E^X(x)E^Y(y)|\psi\rangle = 0,
\end{aligned}$$

so that Eq. (8) concludes $X\psi = Y\psi$.

Busch, Heinonen, and Lahti [9] showed that the condition $X\psi = Y\psi$ does not imply that X and Y are identically distributed. Moreover, we have shown in Section 2 that this happens even for unitarily equivalent observables X and Y. Thus, the condition $X\psi = Y\psi$ does not sufficiently characterize the perfect correlation, even if X and Y have the same spectrum. However, for jointly distributed X and Y, the condition $X\psi = Y\psi$ implies their perfect correlation. To show this, suppose that $X\psi = Y\psi$ and X and Y are jointly distributed in ψ. Then, we have $\langle\psi|E^X(x)E^Y(y)|\psi\rangle \geqslant 0$, and from Eq. (8) we have $(x-y)^2\langle\psi|E^X(x)E^Y(y)|\psi\rangle = 0$ for any $x, y \in \mathbf{R}$. Thus, we have $\langle\psi|E^X(x)E^Y(y)|\psi\rangle = 0$ if $x \neq y$, and by definition X and Y are perfectly correlated in ψ.

Therefore, we have proven the following theorem.

Theorem 1. *Two observables X and Y are perfectly correlated in state ψ if and only if X and Y are jointly distributed and $X\psi = Y\psi$.*

6. Space of perfectly correlating states

Suppose that X and Y are perfectly correlated in ψ. It is natural to ask what states other than ψ have this property. Since X and Y intuitively have the same value in ψ, if we have obtained the result $X = x$ in measuring X without disturbing X and Y, we can also expect to have both $X = x$ and $Y = x$ in the state just after the above measurement. Thus, it is natural to expect that X and Y are perfectly correlated also in the state $E^X(x)\psi/\|E^X(x)\psi\|$ obtained by the above X measurement, and by linearity we can also expect that the state $f(X)\psi/\|f(X)\psi\|$ has this property.

In order to characterize all the states of the form $f(X)\psi/\|f(X)\psi\|$, we introduce the following terminology. The *cyclic subspace* spanned by an observable X and a state ψ is the subspace $\mathcal{C}(X,\psi)$ spanned by $X^n\psi$ for any $n = 0, 1, 2, \ldots$. It is easy to see that $\mathcal{C}(X,\psi)$ is the smallest X invariant subspace of $\mathcal{H}$ including ψ. Denote by $\mathcal{C}_1(X,\psi)$ the unit sphere of $\mathcal{C}(X,\psi)$. Denote by $P_{X,\psi}$ the projection of $\mathcal{H}$ onto $\mathcal{C}(X,\psi)$. Then, we have $f(X)P_{X,\psi} = P_{X,\psi}f(X) = P_{X,\psi}f(X)P_{X,\psi}$ for any function f. Now, we have the following theorem.

Theorem 2. *For any two observables X and Y and any state ψ, the following conditions are equivalent.*

(i) *Observables X and Y are perfectly correlated in state ψ.*
(ii) *Observables X and Y are perfectly correlated in any state $\phi \in \mathcal{C}_1(X,\psi)$.*
(iii) *$f(X)\psi = f(Y)\psi$ for any function f.*
(iv) *$f(X)P_{X,\psi} = f(Y)P_{X,\psi}$.*
(v) *$XP_{X,\psi} = YP_{X,\psi}$.*

Proof. Suppose that condition (i) holds. By the similar computations as before, we have $\|f(X)\psi - f(Y)\psi\|^2 = 0$, and hence, the implication (i) $\Rightarrow$ (iii) follows. Suppose that condition (iii) holds. Then, we have $f(X)g(X)\psi = f(Y)g(Y)\psi = f(Y)g(X)\psi$ for any f and g. Since every $\phi \in \mathcal{C}(X,\psi)$ is of the form $\phi = g(X)\psi$ for some g, we have $f(X)P_{X,\psi} = g(Y)P_{X,\psi}$. Thus, the implication (iii) $\Rightarrow$ (iv) follows. The implication (iv) $\Rightarrow$ (v) is obvious. Suppose that condition (v) holds. Let $P = P_{X,\psi}$. Since X leaves $\mathcal{C}(X,\psi)$ invariant, so does Y. Thus, the spectral projections of YP and XP on $\mathcal{C}(X,\psi)$ are $E^Y(y)P$ and $E^X(y)P$, respectively, and hence $E^Y(y)P = E^X(y)P$ for any $y \in \mathbf{R}$, so that $E^X(x)E^Y(y)P = E^X(x)E^X(y)P$. Thus, we have $\langle\phi|E^X(x)E^Y(y)|\phi\rangle = 0$, if $x \neq y$, for any $\phi \in \mathcal{C}_1(X,\psi)$. It follows that X and Y are perfectly correlated in any state $\phi \in \mathcal{C}_1(X,\psi)$. Thus, the implication (v) $\Rightarrow$ (ii) has been proven. Since the implication (ii) $\Rightarrow$ (i) is obvious, the proof is completed. □

By the above theorem, observables X and Y are represented on the space $\mathcal{C}(X,\psi)$ by the same operator $XP_{X,\psi} = YP_{X,\psi}$, and hence X and Y are considered to be simultaneously measurable in ψ and to have the identical outcomes. In fact, if one measures X and Y by consecutive projective measurements of X and Y, then by Theorem 2(iv) the joint probability distribution of the two outcomes satisfies

$$\begin{aligned}\left\|E^Y(y)E^X(x)\psi\right\|^2 &= \left\|E^X(y)E^X(x)\psi\right\|^2\\ &= \delta_{x,y}\langle\psi|E^X(x)|\psi\rangle,\end{aligned}$$

and hence the measurement outputs actually show the perfect correlation predicted by the theoretical joint probability distribution (7).

7. Characterization of perfectly correlating states

From the above theorem we have the following important characterization of perfectly correlating states.

Theorem 3. *Two observables X and Y are perfectly correlated in a state ψ if and only if ψ is a superposition of common eigenstates of X and Y with common eigenvalues.*

Proof. Suppose that X and Y are perfectly correlated in a state ψ. Then, $\mathcal{C}(X, \psi)$ is generated by eigenstates of $XP_{X,\psi} = YP_{X,\psi}$. Thus, ψ is a superposition of common eigenstates of X and Y with common eigenvalues. Conversely, suppose that ψ is a superposition of common eigenstates of X and Y with common eigenvalues. Then, the subspace $\mathcal{S}$ generated by those eigenstates is invariant under both X and Y and includes ψ. Thus, $\mathcal{C}(X, \psi) \subset \mathcal{S}$, and $X = Y$ on $\mathcal{C}(X, \psi)$, and hence from Theorem 2(v), we conclude X and Y are perfectly correlated in ψ. □

8. Identically distributed observables

Theorem 2(ii) suggests that perfectly correlated X and Y in ψ are equally distributed in any state in the cyclic subspace spanned by ψ and X. The following theorem shows that the converse is also true.

Theorem 4. *Two observables X and Y are perfectly correlated in state ψ if and only if they are identically distributed in any state ϕ in $\mathcal{C}_1(X, \psi)$.*

Proof. Suppose that X and Y are perfectly correlated in state ψ. From Theorem 2(iv), we have $f(X)\phi = f(Y)\phi$ for any function f and $\phi \in \mathcal{C}(X, \psi)$. Taking f to be $f(y) = \delta_{x,y}$, we have $\langle\phi|E^X(x)|\phi\rangle = \langle\phi|E^Y(x)|\phi\rangle$, so that X and Y are identically distributed for any $\phi \in \mathcal{C}_1(X, \psi)$. Conversely, suppose that X and Y are identically distributed in any state ϕ in $\mathcal{C}_1(X, \psi)$. There is an orthonormal basis $\{|n, \nu\rangle\}$ of $\mathcal{C}(X, \psi)$ consisting of eigenstates of X such that $X|n, \nu\rangle = x_n|n, \nu\rangle$. By the identical distributivity of X and Y in $|n, \nu\rangle$, we have $Y|n, \nu\rangle = x_n|n, \nu\rangle$. Thus, ψ is a superposition of common eigenstates of X and Y with common eigenvalues. We conclude, therefore, from Theorem 3 that X and Y are perfectly correlated in state ψ. □

9. Characterization of precise measurements of observables

Let $\mathbf{A}(\mathbf{x})$ be an apparatus with output variable $\mathbf{x}$ for measuring a system $\mathbf{S}$ described by a Hilbert space $\mathcal{H}$. The measuring process of $\mathbf{A}(\mathbf{x})$ is described by a quadruple $(\mathcal{K}, \xi, U, M)$ consisting of a Hilbert space $\mathcal{K}$ describing the probe $\mathbf{P}$, a state vector ξ in $\mathcal{K}$ describing the state of $\mathbf{P}$ just before the measurement, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ describing the time evolution of the composite system $\mathbf{S} + \mathbf{P}$ during the measuring interaction, and an observable M on $\mathcal{K}$ describing the meter observable [3,4,6–8,10,11]. We assume for simplicity that both $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional. If the measuring interaction turns on from time t to $t + \Delta t$, in the Heisenberg picture with original state $\psi \otimes \xi$ at time t, we write $A(t) = A \otimes I$ and $M(t + \Delta t) = U^\dagger(I \otimes M)U$.

The probability distribution of the output variable $\mathbf{x}$ on the input state ψ is given by

$$\Pr\{\mathbf{x} = x \| \psi\} = \langle\psi \otimes \xi|U^\dagger\big[I \otimes E^M(x)\big]U|\psi \otimes \xi\rangle. \tag{9}$$

Let A be an observable on $\mathcal{H}$. Naturally, we should say that the apparatus $\mathbf{A}(\mathbf{x})$ with measuring process $(\mathcal{K}, \xi, U, M)$ *precisely measures* the value of observable A in state ψ, if the observable $A \otimes I$ and $U^\dagger(I \otimes M)U$ are perfectly correlated in the state $\psi \otimes \xi$. In this case, we can say that the measuring interaction reproduces "the value" taken by A before the measuring interaction; if the observer were to measure $A(t)$ and $M(t + \Delta t)$ jointly then the observer would obtain the same value from each measurement, so that the observer can safely report that his value obtained from observing $M(t + \Delta t)$ is the value obtained from the measurement of $A(t)$. On the other hand, the apparatus $\mathbf{A}(\mathbf{x})$ is said to satisfy the *Born statistical formula* (*BSF*) for A in state ψ if

$$\Pr\{\mathbf{x} = x \| \psi\} = \langle\psi|E^A(x)|\psi\rangle \tag{10}$$

for all $x \in \mathbf{R}$. In this case, we can say at least that the measuring interaction reproduces the probability distribution of A before the measuring interaction.

The relation

$$\Pi(x) = \mathrm{Tr}_{\mathcal{K}}\big[U^{\dagger}\big[I \otimes E^{M}(x)\big]U\big(I \otimes |\xi\rangle\langle\xi|\big)\big] \tag{11}$$

defines the *probability operator valued measure* (*POVM*) $\{\Pi(x) \mid x \in \mathbf{R}\}$ of $\mathbf{A}(\mathbf{x})$, where $\mathrm{Tr}_{\mathcal{K}}$ stands for the partial trace over $\mathcal{K}$. Then, the probability distribution of the output is described by

$$\Pr\{\mathbf{x} = x \| \psi\} = \langle\psi|\Pi(x)|\psi\rangle. \tag{12}$$

We say that a POVM $\{\Pi(x) \mid x \in \mathbf{R}\}$ is *perfectly correlated* to an observable A in a state ψ, if

$$\langle\psi|\Pi(x)E^{A}(y)|\psi\rangle = 0 \tag{13}$$

for any $x, y \in \mathbf{R}$ with $x \neq y$. Then, the following theorem characterizes precise measurements of the value of an observable in a given state.

Theorem 5. *Let* $\mathbf{A}(\mathbf{x})$ *be an apparatus with measuring process* $(\mathcal{K}, \xi, U, M)$ *and POVM* $\{\Pi(x) \mid x \in \mathbf{R}\}$. *Then, for any observable A and state* ψ, *the following conditions are all equivalent.*

- (i) $\mathbf{A}(\mathbf{x})$ *precisely measures* A *in* ψ.
- (ii) *The POVM* $\{\Pi(x) \mid x \in \mathbf{R}\}$ *is perfectly correlated to* A *in* ψ.
- (iii) $\mathbf{A}(\mathbf{x})$ *satisfies the BSF for* A *in any* $\phi \in \mathcal{C}_1(A, \psi)$.
- (iv) $\Pi(x)P_{A,\psi} = E^{A}(x)P_{A,\psi}$ *for any* $x \in \mathbf{R}$.

Proof. The equivalence between conditions (i) and (ii) follows immediately from the relation

$$\langle\psi \otimes \xi|\big[E^{A}(x) \otimes I\big]U^{\dagger}\big[I \otimes E^{M}(y)\big]U|\psi \otimes \xi\rangle$$
$$= \langle\psi|E^{A}(x)\Pi(y)|\psi\rangle. \tag{14}$$

We easily obtain the relations

$$\mathcal{C}(A \otimes I, \psi \otimes \xi) = \mathcal{C}(A, \psi) \otimes \mathbf{C}\xi, \tag{15}$$

$$P_{A\otimes I,\psi\otimes\xi} = P_{A,\psi} \otimes |\xi\rangle\langle\xi|. \tag{16}$$

From the above relations, the equivalence of conditions (i) and (iii) follows from Theorem 4. Assume that condition (i) holds. By Theorem 2, condition (i) is equivalent to the relation

$$U^{\dagger}\big[I \otimes E^{M}(x)\big]UP_{A\otimes I,\psi\otimes\xi}$$
$$= \big[E^{A}(x) \otimes I\big]P_{A\otimes I,\psi\otimes\xi} \tag{17}$$

for any $x \in \mathbf{R}$. Then, $U^{\dagger}[I \otimes E^{M}(x)]U$ commutes with $P_{A\otimes I,\psi\otimes\xi}$, so that from Eq. (16) we have

$$U^{\dagger}\big[I \otimes E^{M}(x)\big]UP_{A\otimes I,\psi\otimes\xi}$$
$$= \Pi(x)P_{A,\psi} \otimes |\xi\rangle\langle\xi|. \tag{18}$$

Thus, Eq. (17) implies the relation

$$\Pi(x)P_{A,\psi} \otimes |\xi\rangle\langle\xi| = E^{A}(x)P_{A,\psi} \otimes |\xi\rangle\langle\xi|, \tag{19}$$

so that we have condition (iv). Conversely, it is now easy to see that condition (iv) implies Eq. (17). Thus, condition (i) and condition (iv) are equivalent. □

The above theorem shows that whether an apparatus precisely measures the value of an observable in a given state is determined solely by the corresponding POVM. In the conventional approach, the apparatus $\mathbf{A}(\mathbf{x})$ is said to *precisely measure* the "observable" A, if it satisfies the BSF for A in *every* state ψ of the system **S** [2–4,6–8]. It is well known that $\mathbf{A}(\mathbf{x})$ precisely measures A if and only if $\Pi(x) = E^{A}(x)$ for all $x \in \mathbf{R}$. By Theorem 5, $\mathbf{A}(\mathbf{x})$ satisfies the BSF for A in *every* state ψ of the measured system if and only if the meter observable and the measured observable are perfectly correlated in any input state. Thus, we have justified the conventional definition by having shown that every precise measurement of "observable" A reproduces not only the probability distribution but also the value taken by A before the measurement.

10. von Neumann's model of measurement

It was shown by von Neumann [5] that a measurement of an observable

$$A = \sum_{n} a_n|\phi_n\rangle\langle\phi_n| \tag{20}$$

on $\mathcal{H}$ with eigenvalues $a_0, a_1, \ldots$ and an orthonormal basis of eigenvectors $\phi_0, \phi_1, \ldots$ can be realized by a unitary operator U on the tensor product $\mathcal{H} \otimes \mathcal{K}$ with another separable Hilbert space $\mathcal{K}$ with orthonormal basis $\{\xi_n\}$ such that

$$U(\phi_n \otimes \xi) = \phi_n \otimes \xi_n, \tag{21}$$

where ξ is an arbitrary vector state in $\mathcal{K}$. Let

$$M = \sum_{n} a_n|\xi_n\rangle\langle\xi_n| \tag{22}$$

be an observable on $\mathcal{K}$ called the meter. von Neumann's model defines an apparatus $\mathbf{A}(\mathbf{x})$ with measuring process $(\mathcal{K}, \xi, U, M)$.

Let us suppose that the initial state of the system is given by an arbitrary state vector $\psi = \sum_n c_n \phi_n$. Then, it follows from the linearity of U we have

$$U(\psi \otimes \xi) = \sum_n c_n \phi_n \otimes \xi_n. \tag{23}$$

The conventional explanation as to why this transformation can be regarded as a measurement is as follows; symbols are adapted to the present context in the quote below. "In the state (23), obtained by the measurement, there is a statistical correlation between the state of the object and that of the apparatus: the simultaneous measurement on the system—object-plus-apparatus—of the two quantities, one of which is the originally measured quantity of the object and the second the position of the pointer of the apparatus, always leads to concordant results. As a result, one of these measurements is unnecessary: the state of the object can be ascertained by an observation on the apparatus. This is a consequence of the special form of the state vector (23), on not containing any $\phi_m \otimes \xi_n$ term with $n \neq m$ [12]." "The equations of motion permit the description of the process whereby the state of the object is mirrored by the state of an apparatus. The problem of a measurement on the object is thereby transformed into the problem of an observation on the apparatus [12]."

The above explanation correctly points out the existence of the statistical correlation between the measured observable A and the meter observable M in the state (23). However, this is not the statistical correlation between the measured observable before the interaction and the meter observable after the interaction, but that between those observables after the interaction. Thus, the above statistical correlation does not even ensure that the probability distribution of the measured observable before the interaction is reproduced by the observation of the meter observable after the interaction.

The role of the measuring interaction described by U should be to make the following two correlations: (i) the correlation between the measured observable A *before* the interaction and the meter M *after* the interaction, and (ii) the correlation between the meter M *after* the interaction and the measured observable A *after* the interaction. The first correlation is required by the *value reproducing requirement* that the interaction transfers the value of the measured observable A before the interaction to the value of the meter M after the interaction. The second correlation is required by the *repeatability hypothesis* that if the meter observable M has the value a_n after the interaction, then the observable A also have the same value a_n after the interaction so that the second measurement of A after the interaction reproduce the same value of the meter of the first measurement of A.

Now, we shall show that those requirements are actually satisfied. Let $\eta_0, \eta_1, \ldots$ be an orthonormal basis of $\mathcal{H}$ such that $\eta_0 = \xi$, namely an orthonormal basis extending $\{\xi\}$. Let $\Psi_{n,m}$ be a unit vector in $\mathcal{H}$ defined by $\Psi_{n,m} = U^\dagger(\phi_n \otimes \xi_m)$ for any n, m. Then, we have $\Psi_{n,n} = \phi_n \otimes \xi$ and the family $\{\Psi_{n,m}\}$ is an orthonormal basis of $\mathcal{H}$. By simple calculations, we have

$$A \otimes I = A \otimes |\xi\rangle\langle\xi| + \sum_{m \neq 0} A \otimes |\eta_m\rangle\langle\eta_m|, \tag{24}$$

$$U^\dagger(A \otimes I)U = A \otimes |\xi\rangle\langle\xi| + \sum_{n \neq m} a_n |\Psi_{n,m}\rangle\langle\Psi_{n,m}|, \tag{25}$$

$$U^\dagger(I \otimes M)U = A \otimes |\xi\rangle\langle\xi| + \sum_{n \neq m} a_m |\Psi_{n,m}\rangle\langle\Psi_{n,m}|, \tag{26}$$

where $\sum_{n \neq m}$ stands for the summation over all n, m with $n \neq m$. By the above relations it is now obvious that $A \otimes I = U^\dagger(A \otimes I)U = U^\dagger(I \otimes M)U$ on their common invariant subspace $\mathcal{H} \otimes [\xi]$, so that those three observables are perfectly correlated in the state $\psi \otimes \xi$ for every state vector ψ in $\mathcal{H}$. Therefore, von Neumann's model $(\mathcal{K}, \xi, U, M)$ satisfies both the value reproducing requirement and the repeatability hypothesis.

11. Concluding remarks

In this Letter, we have introduced the notion of perfect correlation between noncommuting observables and explored its basic properties. This notion is applied to characterizing the precise measurement of the value of an observable in a given state and justifies the conventional definition of precise measurement of an observable formulated independently of the input state. Although this Letter has focused on the finite level systems, the theory for the general case can be developed with analogous results under the definition that observables X and Y are *perfectly correlated* in state ψ, if

$$\langle\psi|E^X(\Delta)E^Y(\Gamma)|\psi\rangle = 0 \tag{27}$$

for any mutually disjoint Borel sets Δ and Γ, where E^X and E^Y are the spectral measures of X and Y, respectively; the detail will be discussed in a forthcoming paper.

The notion of perfect correlation is not restricted to the problem of measurement, but also has broad applications in foundations on quantum mechanics [1] and quantum information theory [2]. Those applications will be discussed elsewhere.

Acknowledgements

The author thanks Julio Gea-Banacloche for his helpful suggestions for revising the original manuscript. This work was supported by the SCOPE project of the MPHPT of Japan, by the CREST project of the JST, and by the Grant-in-Aid for Scientific Research of the JSPS.

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