

# THE ASYMPTOTIC BEHAVIOUR OF MASTER EQUATIONS

(Quantum Markov Semigroups)

Franco Fagnola

fagnola@dima.unige.it

WORKSHOP ON QUANTUM OPEN  
SYSTEMS

**Pavia, October 2004**

1. Quantum Markov (Dynamical) Semigroups
2. Recurrent and transient projection
3. Irreducible components
4. Uniqueness of invariant states
5. Convergence to invariant states
6. Exponential ergodicity
7. Generic generator ...
8. 2-photon absorption and emission process

## Quantum (Markov) semigroup

$\mathcal{A}$  von Neumann algebra      (  $\mathcal{A} = (\mathcal{A}_*)^*$  )

$$\mathcal{T}_t : \mathcal{A} \rightarrow \mathcal{A}, \quad t \geq 0$$

i) (semigroup)

$$\mathcal{T}_0(a) = a, \quad \mathcal{T}_t(\mathcal{T}_s(a)) = \mathcal{T}_{t+s}(a),$$

ii) (positive)       $a \geq 0 \Rightarrow \mathcal{T}_t(a) \geq 0,$

iii) (unital)       $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$

iv) (normal)       $a_\alpha \uparrow a \Rightarrow \mathcal{T}_t(a_\alpha) \uparrow \mathcal{T}_t(a),$

v) ( $w^*$  continuous)       $t \rightarrow \mathcal{T}_t(a) \quad \forall a$

$\mathcal{T}$  is subMarkov if  $\mathcal{T}_t(\mathbb{1}) \leq \mathbb{1}$

*Classical System.*

$$\mathcal{A} = L^\infty(E, \mathcal{E}, \mu), \quad \mathcal{A}_* = L^1(E, \mathcal{E}, \mu).$$

*Quantum Open System (typical)*

$$\mathcal{A} = \mathcal{B}(\mathfrak{h}), \quad \mathcal{A}_* = L^1(\mathfrak{h}) \text{ (trace class).}$$

Often positivity ii) is replaced by the stronger

sp) (strongly positive or Schwarz)

$$\mathcal{T}_t(a^*)\mathcal{T}_t(a) \leq \mathcal{T}_t(a^*a) \quad \forall a \in \mathcal{A}$$

cp) (complete positivity)

$$\forall t \geq 0, \forall n \in \mathbb{N}, a_{jk} \in \mathcal{A} \quad (1 \leq j, k \leq n)$$

$$\begin{pmatrix} a_{11} & \cdot & a_{1n} \\ \cdots & \cdot & \cdots \\ a_{n1} & \cdot & a_{nn} \end{pmatrix} \geq 0, \Rightarrow \begin{pmatrix} \mathcal{T}_t(a_{11}) & \cdot & \mathcal{T}_t(a_{1n}) \\ \cdots & \cdot & \cdots \\ \mathcal{T}_t(a_{n1}) & \cdot & \mathcal{T}_t(a_{nn}) \end{pmatrix} \geq 0$$

$(E, \mathcal{E}, \mu)$  ( $\mu$   $\sigma$ -finite) measure space

$$\left( \begin{array}{l} \text{Markov process} \\ \text{values in } E \end{array} \right) \approx \left( \begin{array}{l} \text{Markov semigroup} \\ \text{on } L^\infty(E, \mathcal{E}, \mu) \end{array} \right)$$

$L^\infty(\dots)$  commutative von Neumann algebra

$\mathcal{A}$  von Neumann algebra

$$\left( \begin{array}{l} \text{Q-Markov proc.} \\ \text{values in } \mathcal{A} \end{array} \right) \approx \left( \begin{array}{l} \text{Q-M. semigroup} \\ \text{on } \mathcal{A} \end{array} \right)$$

Problems:

- the Q-M. process goes to  $\infty$  (fast/slow) ?
- or wanders in a “bounded” region ?
- what about invariant states ?
- convergence to an invariant state ?

## Classical Markov chain (finite)

Blocks of the transition matrix  $T$

$$\begin{array}{lll} \text{transient states} & S & \rightarrow \\ \text{recurrent class} & R_1 & \rightarrow \\ \text{recurrent class} & R_2 & \rightarrow \end{array} \left( \begin{array}{c|c|c} * & * & * \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array} \right)$$

$\mu$  initial distribution

$$\frac{1}{n} \sum_{k=1}^n T^{*k} \mu \xrightarrow{n \rightarrow \infty} \nu$$

- $\text{supp}(\nu) \subseteq R_1 \cup R_2$ ,
- if  $\text{supp}(\mu) \subseteq R_k \Rightarrow \text{supp}(\nu) \subseteq R_k \quad \forall k = 1, 2$ ,
- $T1_{R_k} \geq 1_{R_k} \quad (\forall k = 1, 2)$  and  $T1_S \leq 1_S$ ,
- $T^n 1_S \downarrow 0$  (for  $n \rightarrow \infty$ ) and  $\sum_n T^n 1_S < \infty$ .

$T$   $d \times d$  stochastic matrix acts on

$$\mathcal{A} = \ell^\infty(\{1, \dots, d\}; \mathbb{C}^d) \text{ as}$$

$$(Tf)(j) = \sum_k T_{jk} f(k).$$

$T$  is called *irreducible* if

$$\forall i, j \in \{1, \dots, d\}, \exists n > 0 \text{ s.t. } (T^n)_{ij} > 0$$

Let  $\mathcal{A}_I = \{f \in \mathcal{A} \mid f(k) = 0, \forall k \notin I\}$

*The following are equivalent:*

1.  $T$  is irreducible,
2. there exists no **proper** subset  $I$  of  $\{1, \dots, d\}$  such that  $T(\mathcal{A}_I) \subseteq \mathcal{A}_I$ .

$\mathcal{A}_I$ 's are all the *hereditary* subalgebras of  $\mathcal{A}$

i.e.  $\mathcal{B} = p\mathcal{A}p$  for  $p$  projection in  $\mathcal{A}$

Let  $p = 1_I$ . Then (a)  $\Leftrightarrow$  (b)

$$(a) \quad T(\mathcal{A}_I) \subseteq \mathcal{A}_I$$

$$(b) \quad Tp \leq p$$

**Proposition.**  $\Phi$  positive linear on  $\mathcal{A}$  with  $\Phi(\mathbb{1}) \leq \mathbb{1}$  and let  $p \in \mathcal{A}$  be a projection. Then

$$\Phi(p\mathcal{A}p) \subseteq p\mathcal{A}p \quad \Leftrightarrow \quad \Phi(p) \leq p.$$

A projection  $p$  s.t.  $\Phi(p) \leq p$  is *superharmonic*

**Definition.**  $\mathcal{T}$  is *irreducible* iff  $\mathcal{T}_t(p) \leq p \quad \forall t \geq 0$  implies either  $p = \mathbb{1}$  or  $p = 0$ .

**Goal.**

Decompose QMS into irreducible "sub"-QMS.

## Fast recurrent projection

$\mathcal{T}$  QMS on a vN algebra  $\mathcal{A}$

The support projection  $p_\omega$  of a normal invariant state  $\omega$  is subharmonic  $\mathcal{T}_t(p_\omega) \geq p_\omega \quad \forall t \geq 0$ .

**Prop.** Let  $p_\alpha$  be subharmonic projections. Then  $p := \sup_\alpha p_\alpha$  is a subharmonic projection.

$\mathcal{S}$  normal invariant states

**Defn.** *Fast recurrent projection*

$$p_R := \sup_{\omega \in \mathcal{S}} p_\omega, \quad (p_\omega \text{ support of } \omega).$$

Subharmonicity  $\mathcal{T}_t(p_R) \geq p_R \Rightarrow (\mathcal{T}_* \text{ predual sgr})$

$$\boxed{\text{supp}(\mathcal{T}_{*t}(\sigma)) \subseteq p_R}$$

for all state  $\sigma$  with  $\text{supp}(\sigma) \subseteq p_R$ .

## Fast recurrent part of $\mathcal{T}$

QMS on  $p_R \mathcal{A} p_R$        $\mathcal{T}_t^R := \left( \mathcal{T}_{*t} \Big|_{p_R \mathcal{A} p_R} \right)^*$

$$\mathcal{T}_t^R(b) = p_R \mathcal{T}_t(b) p_R \quad \forall b = p_R a p_R, \quad a \in \mathcal{A}$$

**Prop.**  $\forall a \in \mathcal{A}$  there exists

$$w^* - \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t p_R \mathcal{T}_s(a) p_R ds$$

and defines a conditional expectation

$$\mathcal{E} : \mathcal{A} \rightarrow \mathcal{F}(\mathcal{T}^R).$$

( $\mathcal{F}(\mathcal{T}^R)$  fixed points of  $\mathcal{T}^R$ ).

Decomposition of  $\mathcal{T}^R$

recurrent class	$p_1$	$\rightarrow$	$\left( \begin{array}{c c c c} * & 0 & 0 & \dots \\ \hline 0 & * & 0 & \dots \\ \hline 0 & 0 & * & \dots \\ \hline \dots & \dots & \dots & \dots \end{array} \right)$
recurrent class	$p_2$	$\rightarrow$	
recurrent class	$p_3$	$\rightarrow$	
recurrent class	$\dots$	$\rightarrow$	

**Thm.** Suppose  $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{h})$   $\sigma$ -finite and

either  $\mathcal{F}(\mathcal{F}_*)$  is finite dimensional,

or  $\mathcal{A}$  is abelian

Then there exists a sequence  $(p_n)_{n \geq 1}$  of orthogonal projections in  $\mathfrak{h}$  such that

1.

$$p_R = \sum_{n \geq 1} p_n, \quad \mathcal{T}_t^R(p_n) = p_n \quad \forall n,$$

2. the reduced semigroup

$$\mathcal{T}_t^{p_n}(x) := p_n \mathcal{T}_t(x) p_n$$

(on  $p_n \mathcal{A} p_n$ ) is irreducible.

$\forall n$   $\mathcal{T}^{p_n}$  (QMS on  $p_n \mathcal{A} p_n$ ) has a *unique* faithful normal invariant state.

Form potential ( $\mathcal{A}$  acts on a Hilbert space  $\mathfrak{h}$ )

$$\mathcal{U}_f(x)[u] = \int_0^\infty \langle u, \mathcal{T}_t(x)u \rangle dt \quad x \in \mathcal{A}_+, \quad u \in \mathfrak{h}.$$

**Prop.** Projections onto  $\overline{\{u \in \mathfrak{h} \mid \mathcal{U}_f(x)[u] < \infty\}}$  and  $\{u \in \mathfrak{h} \mid \mathcal{U}_f(x)[u] = 0\}$  are in  $\mathcal{A}$  and are *subharmonic*.

If  $\mathcal{U}_f(x)$  is densely defined we find a *selfadjoint* operator  $\mathcal{U}(x)$  representing the form  $\mathcal{U}_f(x)$ .

$$\mathcal{A}_{\text{int}} := \left\{ x \in \mathcal{A}_+ \mid \mathcal{U}(x) \text{ bounded} \right\}$$

$$\mathcal{P}_{\text{tr}} := \{ p \text{ proj} \in \mathcal{A} \mid \exists x \in \mathcal{A}_{\text{int}} \text{ s.t. } p = \text{supp}(\mathcal{U}(x)) \}$$

**Defn.** *Transient projection* associated with  $\mathcal{T}$

$$p_T := \sup_{p \in \mathcal{P}_{\text{tr}}} p$$

**Remark.**  $p_T \leq p_R^\perp$ .

**Thm.** If  $\mathcal{A}$  is  $\sigma$ -finite then

1. there exists an increasing sequence  $(p_n)_{n \geq 1}$ ,  
 $p_n \in \mathcal{A}_{\text{int}} \forall n$  s.t.  $p_T = \sup_n p_n$ ,
2.  $p_T$  is *superharmonic* i.e.  $\mathcal{T}_t(p_T) \leq p_T \forall t$ .

**Rem.** Transient part.  $\mathcal{T}_t^T := \mathcal{T}_t|_{p_T \mathcal{A}_{p_T}}$ .

**Defn.** *slow recurrent projection*

$$p_{R_0} := p_R^\perp - p_T$$

**Defn.** A QMS  $\mathcal{T}$  is called

<i>transient</i>	if	$p_T = \mathbb{1}$ ,
<i>recurrent</i>	if	$p_T = 0$ ,
<i>slow recurrent</i>	if	$p_{R_0} = \mathbb{1}$ ,
<i>fast recurrent</i>	if	$p_R = \mathbb{1}$ .

**Prop.** An irreducible QMS  $\mathcal{T}$  is either transient, slow recurrent or fast recurrent.

Potentials and superharmonic operators.

$$\begin{aligned}\mathcal{T}_t(\mathcal{U}(x)) &= \mathcal{T}_t\left(\int_0^\infty \mathcal{T}_s(x) ds\right) \\ &= \int_t^\infty \mathcal{T}_s(x) ds \leq \mathcal{U}(x)\end{aligned}$$

**Proposition.** *An irreducible QMS  $\mathcal{T}$  is transient if and only if it admits a non-trivial superharmonic operator.*

**Corollary.**  *$\mathcal{A} = \mathcal{B}(\mathfrak{h})$ ,  $\mathfrak{h}$  separable. A transient  $\mathcal{T}$  has no normal invariant state.*

**Rem.**  $\exists \mathcal{U}(p) \Rightarrow \mathcal{T}_t(p) \xrightarrow{t \rightarrow \infty} 0$ .

Indeed,  $\mathcal{U}(p_n)$  bounded implies  $\mathcal{T}_t(p_n) \rightarrow_n 0$ . Thus, if  $\omega(\mathcal{T}_t(a)) = \omega(a) \forall t \geq 0$ , then  $\exists \nu$  s.t.  $\omega(p_n) > 1/2 \forall n > \nu$  and

$$1/2 < \limsup_n \omega(p_n) = \limsup_n \omega(\mathcal{T}_t(p_n)) = 0.$$

$\mathcal{A}$   $\sigma$ -finite ( $\mathcal{T}$  could be 2-positive)

**Thm.** The following are equivalent:

**T1**  $\exists x \in \mathcal{A}_+$ , with  $\mathcal{U}(x)$  bounded and  $\mathcal{U}(x) > 0$ ,

**T2**  $\exists$  a positive  $x \in \mathcal{A}$  with  $\mathcal{U}(x) > 0$ ,

**T3**  $\exists$  an increasing sequence of projections  $(p_n)_{n \geq 1}$ , with  $p_n \uparrow \mathbb{1}$  and  $\mathcal{U}(p_n)$  bounded  $\forall n$ .

**Thm.** The following are equivalent:

**R1**  $\forall x \in \mathcal{A}_+$  and  $u \in \mathfrak{h}$  either  $\mathcal{U}_f(x)[u] = +\infty$  or  $\mathcal{U}_f(x)[u] = 0$ ,

**R2**  $\forall$  projection  $p \in \mathcal{A}$  and  $u \in \mathfrak{h}$  either  $\mathcal{U}_f(p)[u] = +\infty$  or  $\mathcal{U}_f(p)[u] = 0$ ,

**Rem.** If  $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ , with  $\mathfrak{h}$  separable, projections above can be taken finite dimensional.

## Asymptotic behaviour

**Defn.** A QMS  $\mathcal{T}$  is:

a) *mean ergodic* if  $\exists$

$$w^* - \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathcal{T}_s(a) ds \quad \forall a \in \mathcal{A}.$$

b) *ergodic* if  $\exists$

$$w^* - \lim_{t \rightarrow \infty} \mathcal{T}_t(a) \quad \forall a \in \mathcal{A}.$$

*Pre dual semigroup*  $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$  : the unique sgr on  $\mathcal{A}_*$  s.t.  $(\mathcal{S}_t)^* = \mathcal{T}_t$ .

**Prop.** Any  $w$ -limit  $\rho$  of the Césaro means

$$\frac{1}{t_\alpha} \int_0^{t_\alpha} \mathcal{S}_s(\sigma) ds \quad (\sigma \in \mathcal{A}_*)$$

is  $\mathcal{S}$ -invariant i.e.  $\mathcal{S}_t(\rho) = \rho \quad \forall t$ .

• *Support projection* of  $\omega \in \mathcal{A}_*$ : smallest proj.  $p \in \mathcal{A}$  s.t.  $\omega(a) = \omega(pa) = \omega(ap) = \omega(pap) \quad \forall a$ .

## $\mathcal{T}$ Schwarz QMS - Q1: $\exists$ invariant state ?

**Rec.** An invariant state exists iff  $\exists$  a non-zero positive  $\omega \in \mathcal{A}_*$  s.t.  $(t^{-1} \int_0^t \mathcal{S}_s(\omega) ds; t \geq 0)$  is  $w$ -relatively compact.

**Thm.** (FF, Rebolledo 2001)  $\mathcal{A} = \mathcal{B}(h)$  ( $h$  sep.). If there exists  $X, Y$  self-adjoint with  $X \geq 0$ ,  $Y$  bounded form below with finite-dimensional spectral projections  $E_Y] - \infty, r]$  s.t.  $\forall t, r \geq 0$

$$\int_0^t \langle u, \mathcal{T}_s(Y \wedge r)u \rangle ds \leq \langle u, Xu \rangle$$

$\forall u \in \text{Dom}(X)$  then  $\mathcal{T}$  has an invariant state.

(Formally eq. to  $\mathcal{L}(X) \leq -Y$ ;  $\mathcal{L}$  generator).

**Thm.** (Emel'yanov, Wolff 2002) Suppose  $\mathcal{A}$  semifinite. If there exists a  $\eta \in \mathcal{A}_*$  s.t.

$$\lim_{t \rightarrow \infty} \left\| \left( \eta - t^{-1} \int_0^t \mathcal{S}_s(\omega) ds \right)_+ \right\| = 0$$

$\forall$  positive  $\omega$  with  $\|\omega\| = 1$  then  $\mathcal{T}$  has an invariant state.

$\mathcal{T}$  Schwarz QMS,  $\rho$  faithful invariant state  
**uniqueness**

$$\mathcal{F}(\mathcal{T}) := \{ a \in \mathcal{A} \mid \mathcal{T}_t(a) = a \}$$

**Prop.**

1.  $\mathcal{F}(\mathcal{T})$  is a vN subalgebra of  $\mathcal{A}$ ,
2.  $\exists \mathcal{E} : \mathcal{A} \rightarrow \mathcal{F}(\mathcal{T})$  (Umegaki) cond. expect.

**Thm.** The following are equivalent

1.  $\rho$  is the unique normal  $\mathcal{T}$ -invariant state,
2.  $\mathcal{F}(\mathcal{T}) = \mathbb{C} \cdot \mathbb{1}$ ,
3.  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma(\mathcal{T}_s(a)) ds = \sigma(\mathbb{1})\rho(a)$   
 $\forall a \in \mathcal{A}, \sigma \in \mathcal{A}_*$ .

$\mathcal{T}$  Schwarz QMS,  $\rho$  faithful invariant state

Q2: is  $\mathcal{T}$  ergodic ?

$$\mathcal{N}(\mathcal{T}) := \{ a \in \mathcal{A} \mid \mathcal{T}_t(a^*)\mathcal{T}_t(a) = \mathcal{T}_t(a^*a), \\ \mathcal{T}_t(a)\mathcal{T}_t(a^*) = \mathcal{T}_t(aa^*) \quad \forall t \geq 0 \}$$

**Ex.**  $\mathcal{T}_t(a) = e^{-itH} a e^{itH}$  ( $H$  s.a.)  $\Rightarrow \mathcal{N}(\mathcal{T}) = \mathcal{A}$

**Thm.** (Frigerio, Verri 1982) If  $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$  then  $\forall a \in \mathcal{A}$  there exists

$$w^* - \lim_{t \rightarrow \infty} \mathcal{T}_t(a)$$

## Remark

- This suff. condition is often necessary.
- $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) \approx$  periph. spectr. of  $\mathcal{T}_t = \{1\}$ .

Next step: conditions on the generator of  $\mathcal{T}$

- $\mathcal{T}_t(a) \leq a$  iff  $\mathcal{L}(a) \leq 0$
- $\mathcal{T}_t(a^*)\mathcal{T}_t(a) = \mathcal{T}_t(a^*a)$  iff  
 $a^*\mathcal{L}(a) + \mathcal{L}(a^*)a = \mathcal{L}(a^*a).$

$$\mathcal{L}(x) = G^*x + \sum_{\ell=1}^{\infty} L_{\ell}^*xL_{\ell} + xG$$

- conditions on  $G, L_1, \dots$  for  $p$  subharmonic,
- relationship between  $\mathcal{F}(\mathcal{T})$  and  $G, L_1, \dots,$
- relationship between  $\mathcal{N}(\mathcal{T})$  and  $G, L_1, \dots,$

## Lindblad's theorem

$\mathcal{A} = \mathcal{B}(\mathfrak{h})$  ( $\mathfrak{h}$  sep.) and  $\mathcal{T}$  completely positive norm-continuous i.e.

$$\lim_{t \rightarrow 0} \sup_{x \in \mathcal{B}(\mathfrak{h}), \|x\| \leq 1} \|\mathcal{T}_t(x) - x\| = 0$$

semigroup with *infinitesimal generator*

$$\mathcal{L} : \mathcal{B}(\mathfrak{h}) \rightarrow \mathcal{B}(\mathfrak{h}), \quad \mathcal{L}(x) = \text{norm} - \lim_{t \rightarrow 0} \frac{\mathcal{T}_t(x) - x}{t}.$$

$$\begin{aligned} \Rightarrow \quad \mathcal{L}(x) &= i[H, x] \\ &+ \frac{1}{2} \sum_{\ell=1}^{\infty} (-L_{\ell}^* L_{\ell} x + 2L_{\ell}^* x L_{\ell} - x L_{\ell}^* L_{\ell}) \end{aligned}$$

where  $i[H, x] = i(Hx - xH)$  and

- $L_{\ell}$ ,  $\ell = 1, 2, \dots$ , and  $H$  are in  $\mathcal{B}(\mathfrak{h})$ ,
- $H = H^*$  and  $\sum_{\ell} L_{\ell}^* L_{\ell}$  converges strongly on  $\mathfrak{h}$ .

Conversely such an  $\mathcal{L}$  generates a norm-cont.  $\mathcal{T}$

**Remark.**

$$\mathcal{T}_t(\mathbb{1}) \leq \mathbb{1} \text{ (resp. } = \mathbb{1}) \text{ iff } \mathcal{L}(\mathbb{1}) \leq 0 \text{ (resp. } = 0).$$

Writing  $G = -\frac{1}{2} \sum_{\ell \geq 1} L_\ell^* L_\ell - iH$  then

$$\mathcal{L}(x) = G^* x + \sum_{\ell \geq 1} L_\ell^* x L_\ell + xG.$$

## Subharmonic projections

$\mathcal{T}_t(p) \geq p$  iff  $\mathcal{L}(p) \geq 0$  iff  $\mathcal{L}(p^\perp) \leq 0$  then

$$\begin{aligned} 0 \geq p\mathcal{L}(p^\perp)p &= pG^*p^\perp p + \sum_{\ell \geq 1} pL_\ell^*p^\perp L_\ell p + pp^\perp Gp \\ &= \sum_{\ell \geq 1} pL_\ell^*p^\perp L_\ell p \end{aligned}$$

then  $p^\perp L_\ell p = 0$  ( $p$  is  $L_\ell$ -invariant  $\forall \ell$ ). Moreover  $\mathcal{L}(p^\perp) \leq 0$  and  $p\mathcal{L}(p^\perp)p = 0$  implies

$$\mathcal{L}(p^\perp)p = 0 \quad \Rightarrow \quad p^\perp Gp = 0.$$

**Prop.** A projection  $p$  is subharmonic iff  $p$  is an invariant subspace for  $G$  and all the  $L'_\ell$ s.

If  $p$  is a maximal subharmonic projection reduce  $\mathcal{T}$  to  $p^\perp \mathcal{A} p^\perp$  and go on ...

## Existence of invariant states

$$\int_0^t \mathcal{T}_s(Y) ds \leq X \quad \Leftrightarrow \quad \mathcal{L}(X) \leq -Y.$$

$\Rightarrow$  Differentiate at  $t = 0$

$$\int_0^t \mathcal{T}_s(Y) ds \leq X - \mathcal{T}_t(X).$$

$$\Leftrightarrow \quad \frac{d}{dt} \left( \mathcal{T}_t(X) - X + \int_0^t \mathcal{T}_s(Y) ds \right) \\ = \mathcal{T}_t(\mathcal{L}(X) + Y) \leq 0.$$

**Thm.** (FF, Rebolledo JMP 2001)  $\mathcal{A} = \mathcal{B}(h)$  (h sep.). If there exists  $X, Y$  self-adjoint with  $X \geq 0$ ,  $Y$  bounded from below with finite-dimensional spectral projections  $E_Y] - \infty, r]$  s.t.  $\mathcal{L}(X) \leq -Y$  then  $\mathcal{T}$  has an invariant state.

(add. assumptions for unbd. Lindblad type  $\mathcal{L}$ )

**Rem.** If  $\mathcal{T}$  is irreducible any normal  $\mathcal{T}$ -invariant state  $\rho$  is *faithful*. (The support projection of  $\rho$  is subharmonic).

$\mathcal{T}$  norm-continuous QMS,  $\mathcal{L}$  Lindblad generator

$\rho$  faithful normal  $\mathcal{T}$ -invariant state

**Thm.**  $\mathcal{F}(\mathcal{T}) = \{H, L_1, L_1^*, L_2, L_2^*, \dots\}'$  and  
 $\mathcal{N}(\mathcal{T}) = \{L_1, L_1^*, L_2, L_2^*, \dots\}'$

**Thm.** (FF, Rebolledo IDAQP '98) (\*). *If*

$$\{H, L_1, L_1^*, L_2, L_2^*, \dots\}' = \{L_1, L_1^*, L_2, L_2^*, \dots\}'$$

*then  $\mathcal{T}$  is ergodic ( $\mathcal{T}_t(a)$   $w^*$ -conv. for  $t \rightarrow \infty$ ).  
If, moreover,  $\mathcal{N}(\mathcal{T}) = \mathcal{O}\mathbb{1}$  then  $\rho$  is the unique  
faithful normal invariant state.*

(\*) Additional assumptions also for non norm-continuous  $\mathcal{T}$

## Example: decay to vacuum

$\mathfrak{h} = \ell^2(\mathbb{N})$  c.o.n. basis  $(e_n)_{n \geq 0}$ . Generator

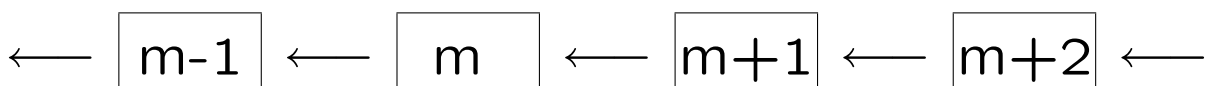
$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{l < m} \gamma_{lm} (\{|e_m\rangle\langle e_m|, x\} - 2|e_m\rangle\langle e_l|x|e_l\rangle\langle e_m|)$$

where,  $H := \sum_{m \geq 0} \varepsilon_m |e_m\rangle\langle e_m|$ ,  $\varepsilon_m \in \mathbb{R}$ ,  $\gamma_{lm} \geq 0$ .

Origin: stochastic limit of  
(reservoir) + (system) + (dypole interaction)

Define the number operator  $N := \sum_m m |e_m\rangle\langle e_m|$ .  
For “good” functions  $f \in \ell^\infty(\mathbb{N}; \mathbb{C})$

$$\begin{aligned} \mathcal{L}(f(N)) &= \sum_m \sum_{l < m} \gamma_{lm} (f(l) - f(m)) |e_m\rangle\langle e_m| \\ &= \sum_{k > 0} \gamma_{N-k, N} (f(N-k) - f(N)) \end{aligned}$$



## Example: decay to vacuum 2

$L_\ell$ 's here are the  $L_{lm} = \gamma_{lm}^{1/2} |e_l\rangle\langle e_m|$  and

$$G = -\frac{1}{2} \left( \sum_m \left( \sum_{l < m} \gamma_{lm} \right) |e_m\rangle\langle e_m| \right) - iH$$

Note that  $[G, N] = 0$  ( $N$  number operator).

### $p$ Subharmonic projection

$$\Rightarrow \gamma_{lm}^{1/2} |p^\perp e_l\rangle\langle p e_m| = 0 \quad \forall l < m$$

$$\Leftrightarrow \text{if } \gamma_{lm} > 0 \text{ either } p e_m = 0 \text{ or } p^\perp e_l = 0$$

$$\Leftrightarrow \text{if } \gamma_{lm} > 0 \text{ either } p^\perp e_m = e_m \text{ or } p e_l = e_l$$

Suppose, e.g.,  $\gamma_{l+1} > 0 \forall l$  then  $p^\perp e_{l+1} = e_{l+1}$  or  $p e_l = e_l$  i.e.  $p$  has the form

$$p = \sum_{j=0}^k |e_j\rangle\langle e_j|, \quad k \in \mathbb{N} \cup \{+\infty\}$$

**unbounded  $L_\ell, H$**

### **Hypothesis H-min**

(i)  $G$  generates a strongly cont. sgr  $(P_t)$  on  $h$ ,

(ii)  $\text{Dom}(L_\ell) \supseteq \text{Dom}(G) \quad \forall \ell \geq 1$ ,

(iii)  $\exists c \in \mathbb{R}$  constant s.t.  $\forall u \in \text{Dom}(G)$

$$2\Re\langle u, Gu \rangle + \sum_{\ell \geq 1} \|L_\ell u\|^2 \leq c\|u\|^2.$$

Define

$$\mathcal{L}(x)[v, u] := \langle v, xGu \rangle + \sum_{\ell} \langle L_\ell v, xL_\ell u \rangle + \langle Gv, xu \rangle$$

**Thm** Under **Hyp. A**  $\exists$  a QDS satisfying

$$\langle v, \mathcal{I}_t(x)u \rangle = \langle v, xu \rangle + \int_0^t \mathcal{L}(\mathcal{I}_s(x))[v, u] ds$$

$\forall u, v \in \text{Dom}(G), \forall x \in \mathcal{B}(h)$ .

The QDS satisfying

$$\langle v, \mathcal{T}_t(x)u \rangle = \langle v, xu \rangle + \int_0^t \mathcal{L}(\mathcal{T}_s(x))[v, u] ds \quad (E)$$

a) not nec. unique, b) might satisfy  $\mathcal{T}_t(\mathbb{1}) < \mathbb{1}$ .

**Prop.** *The following are equivalent:*

1.  $\mathcal{T}$  is the unique solution to (E),
2.  $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ ,
3. the domain of the generator  $\mathcal{L}$  of  $\mathcal{T}$  is  
 $\{x \in \mathcal{B}(\mathfrak{h}) \mid \mathcal{L}(x) \text{ is bounded} \}$ .

**Thm.** (FF, AMC) (Techical assumptions ...)  
*Suppose that  $\exists$  a self-adjoint  $X$  such that:*

$$\sum_{\ell} L_{\ell}^* L_{\ell} \leq X, \quad \mathcal{L}(X) \leq bX.$$

( $b$  constant). Then  $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ .

$\mathfrak{h} = \ell^2(\mathbb{N})$ ,  $a$  annihilation,  $a^\dagger$  creation,  $N$  number

$$\mathcal{L}(x) = G^*x + \sum_{\ell=1}^2 L_\ell^*xL_\ell + xG$$

$$D(G) = D(N^2),$$

$$G = -\frac{\lambda^2}{2}a^2a^{\dagger 2} - \frac{\mu^2}{2}a^{\dagger 2}a^2 - i\omega a^{\dagger 2}a^2$$

$\lambda \geq 0, \mu > 0, \omega \in \mathbb{R}$ .

$$D(L_1) = D(L_2), \quad L_1 = \mu a^2, \quad L_2 = \lambda a^{\dagger 2}.$$

*Regular* if  $\lambda \leq \mu \dots X = N^2, \mathcal{L}(X) \leq bX$ .

*Not regular* if  $\lambda > \mu$

*Subharmonic projections* if  $\lambda > 0$

$$0, p_e = \sum_{k \geq 0} |e_{2k}\rangle\langle e_{2k}|, p_o = \sum_{k \geq 0} |e_{2k+1}\rangle\langle e_{2k+1}|, \mathbb{1}$$

Indeed  $\mathcal{T}_t(p_e) = p_e, \mathcal{T}_t(p_o) = p_o$ .

$$\mathcal{A}_e := p_e \mathcal{A} p_e, \quad \mathcal{A}_o := p_o \mathcal{A} p_o,$$

$$\mathcal{T}_t(\mathcal{A}_e) \subseteq \mathcal{A}_e, \quad \mathcal{T}_t(\mathcal{A}_o) \subseteq \mathcal{A}_o.$$

$$\mathcal{T}_t^e = \mathcal{T}_t|_{\mathcal{A}_e}, \quad \mathcal{T}_t^o = \mathcal{T}_t|_{\mathcal{A}_o}$$

Faithful invariant states ( $\nu = \lambda/\mu < 1$ )

$$\mathcal{T}_t^e \quad \rho_e = (1 - \nu^2) \sum_{k=0}^{\infty} \nu^{2k} |e_{2k}\rangle \langle e_{2k}|$$

$$\mathcal{T}_t^o \quad \rho_o = (1 - \nu^2) \sum_{k=0}^{\infty} \nu^{2k} |e_{2k+1}\rangle \langle e_{2k+1}|$$

$$\mathcal{F}(\mathcal{T}^e) = \mathcal{N}(\mathcal{T}^e) = \mathcal{C} \cdot \mathbb{1} = \mathcal{F}(\mathcal{T}^o) = \mathcal{F}(\mathcal{T}^o)$$

Then  $\rho_e$  (resp.  $\rho_o$ ) is the unique invariant state of  $\mathcal{T}^e$  (resp.  $\mathcal{T}^o$ ) and, for any even state  $\sigma_e$  on  $\mathcal{A}_e$  (resp. odd state  $\sigma_o$  on  $\mathcal{A}_o$ ), we have

$$\lim_{t \rightarrow \infty} \mathcal{T}_{*t}^e(\sigma_e) = \rho_e, \quad \lim_{t \rightarrow \infty} \mathcal{T}_{*t}^o(\sigma_o) = \rho_o.$$

$$\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) = \text{span}\{p_e, p_o\}.$$

**Thm.** For  $\lambda > 0$  all the  $\mathcal{T}$ -invariant states are

$$\alpha\rho_e + (1 - \alpha)\rho_o, \quad \alpha \in [0, 1]$$

and, for all state  $\sigma$  on  $\mathcal{A}$ ,

$$w - \lim_{t \rightarrow \infty} \mathcal{T}_{*t}(\sigma) = \text{tr}(\rho_e p_e)\rho_e + \text{tr}(\rho_o p_o)\rho_o.$$

If  $\lambda = 0$  all subharmonic proj. are generated by

$$\sum_{k=0}^n |e_{2k}\rangle\langle e_{2k}|, \quad \sum_{k=0}^m |e_{2k+1}\rangle\langle e_{2k+1}|,$$

( $n, m \in \mathbb{N} \cup \{+\infty\}$ ). All the invariant states are

$$\sum_{0 \leq j, k \leq 2} \sigma_{jk} |e_j\rangle\langle e_k|,$$

$$\sigma_{00}, \sigma_{11} \geq 0, \sigma_{00} + \sigma_{11} = 1, \sigma_{01} = \bar{\sigma}_{10}, |\sigma_{01}|^2 \leq \sigma_{00}\sigma_{11}.$$

Moreover

$$w - \lim_{t \rightarrow \infty} \mathcal{T}_{*t}(\eta)$$

exist ... exponential speed.

## References

- F. Fagnola, R. Quezada: Two-photon absorption and emission process. Preprint 2004.
- F. Fagnola, R. Rebolledo: On the existence of invariant states for quantum dynamical semigroups. *J. Math. Phys.* **42** (2001), no. 3, 1296–1308.
- F. Fagnola, R. Rebolledo: Subharmonic projections for a quantum Markov semigroup. *J. Math. Phys.* **43** (2002), no. 2, 1074–1082.
- F. Fagnola, R. Rebolledo: Transience and Recurrence of Quantum Markov Semigroups. *Probab. Theory Related Fields* **126** (2003) n.2, 289-306.
- V. Umanità: Classification and decomposition of QMS. Preprint 2004.